Short Proofs of Summation and Transformation Formulas for Basic Hypergeometric Series*

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Abstract. We show that several terminating summation and transformation formulas for basic hypergeometric series can be proved in a straightforward way. Along the same line, new finite forms of Jacobi's triple product identity and Watson's quintuple product identity are also proved.

Keywords: Jackson's ${}_8\phi_7$ summation, Watson's ${}_8\phi_7$ transformation, Bailey's ${}_{10}\phi_9$ transformation, Singh's quadratic transformation, Jacobi's triple product identity, Watson's quintuple product identity

1 Introduction

We follow the standard notation for q-series and basic hypergeometric series in [7]. The *q-shifted factorial* is defined by

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),\quad \text{and}\quad (a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty},\quad \text{for}\quad n\in\mathbb{Z}.$$

As usual, we employ the abbreviated notation

$$(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,\quad \text{for}\quad n=\infty\quad \text{or}\quad n\in\mathbb{Z}.$$

The *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\ldots,a_{r+1}\\ b_1,b_2,\ldots,b_r\end{matrix};q,\ z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{r+1};q)_k z^k}{(q,b_1,b_2,\ldots,b_r;q)_k}.$$

An ${}_{r+1}\phi_r$ series is called *well-poised* if $a_1q=a_2b_1=\cdots=a_{r+1}b_r$ and *very-well-poised* if it is well-poised and $a_2=-a_3=q\sqrt{a_1}$.

The starting point of this paper is the observation that the kth term of a well-poised hypergeometric series

$$F_k(a_1,a_2,\ldots,a_{r+1};q,z):=\frac{(a_1,\ldots,a_{r+1};q)_k}{(q,a_1q/a_2,\ldots,a_1q/a_{r+1};q)_k}z^k$$

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satisfies the following contiguous relations:

$$\begin{aligned} F_k(a_1, a_2, \ldots, a_r q, a_{r+1}; q, z) &- F_k(a_1, a_2, \ldots, a_r, a_{r+1} q; q, z) \\ &= \alpha F_{k-1}(a_1 q^2, a_2 q, \ldots, a_{r+1} q; q, z), \end{aligned} \tag{1.1}$$

$$\begin{aligned} F_k(a_1, a_2, \ldots, a_r, a_{r+1}; q, qz) &- F_k(a_1, a_2, \ldots, a_r, a_{r+1} q; q, z) \\ &= \beta F_{k-1}(a_1 q^2, a_2 q, \ldots, a_{r+1} q; q, z), \end{aligned} \tag{1.2}$$

where

$$\alpha = \frac{(a_r - a_{r+1})(1 - a_1/a_r a_{r+1})(1 - a_1)(1 - a_1 q)(1 - a_2) \cdots (1 - a_{r-1}) z}{(1 - a_1/a_r)(1 - a_1/a_{r+1})(1 - a_1 q/a_2) \cdots (1 - a_1 q/a_{r+1})},$$

$$\beta = -\frac{(1 - a_1)(1 - a_1 q)(1 - a_2) \cdots (1 - a_r) z}{(1 - a_1/a_{r+1})(1 - a_1 q/a_2) \cdots (1 - a_1 q/a_{r+1})}.$$

Indeed, (1.1) and (1.2) correspond respectively to the following trivial identities:

$$\frac{(1 - bx)(1 - ax/b)}{(1 - b)(1 - a/b)} - \frac{(1 - cx)(1 - ax/c)}{(1 - c)(1 - a/c)} = \frac{(b - c)(1 - a/bc)(1 - x)(1 - ax)}{(1 - b)(1 - c)(1 - a/b)(1 - a/c)},$$

$$x - \frac{(1 - cx)(1 - ax/c)}{(1 - c)(1 - a/c)} = -\frac{(1 - x)(1 - ax)}{(1 - c)(1 - a/c)},$$

with $a = a_1$, $b = a_r$, $c = a_{r+1}$ and $x = q^k$.

In this paper we shall give one-line human proofs of several important well-poised q-series identities based on the above contiguous relations. This is a continuation of our previous works [8, 9], where some simpler q-identities were proved by this technique.

Given a summation formula

$$\sum_{k=0}^{n} F_{n,k}(a_1, \ldots, a_s) = S_n(a_1, \ldots, a_s), \tag{1.3}$$

where $F_{n,k}(a_1, \ldots, a_s) = 0$ if $k < 0$ or $k > n$, if one can show that the summand $F_{n,k}(a_1, \ldots, a_s)$ satisfies the following recurrence relation:

$$F_{n,k}(a_1, \ldots, a_s) - F_{n-1,k}(a_1, \ldots, a_s) = \gamma_n F_{n-1,k-1}(b_1, \ldots, b_s) \tag{1.4}$$

for some parameters $b_1, \ldots, b_s$, where γ_n is independent of k, then the proof of the identity (1.3) is completed by induction if one can show that $S_n(a_1, \ldots, a_s)$ satisfies the following recurrence relation

$$S_n(a_1, \ldots, a_s) - S_{n-1}(a_1, \ldots, a_s) = \gamma_n S_{n-1}(b_1, \ldots, b_s). \tag{1.5}$$

If $S_n(a_1, \ldots, a_s)$ appears as a *closed form* as in Jackson's ${}_8\phi_7$ summation formula (2.1), then the verification of recursion (1.5) is routine. If $S_n(a_1, \ldots, a_s)$ appears as a sum of n terms, i.e.,

$$S_n(a_1, \ldots, a_s) = \sum_{k=0}^{n} G_{n,k}(a_1, \ldots, a_s), \tag{1.6}$$

where $G_{n,k}(a_1,\dots,a_s)=0$ if $k<0$ or $k>n$, we may apply q-Gosper's algorithm [12, p. 75] to find a sequence $H_{n,k}(a_1,\dots,a_s)$ of closed forms such that

$$G_{n,k}(a_1,\dots,a_s)-G_{n-1,k}(a_1,\dots,a_s)-\gamma_n G_{n-1,k-1}(b_1,\dots,b_s)=H_{n,k}-H_{n,k-1} \tag{1.7}$$

and $H_{n,n}=H_{n,-1}=0$, then by telescoping we get (1.5). Clearly, we have

$$H_{n,k}=\sum_{j=0}^{k}\left(G_{n,j}(a_1,\dots,a_s)-G_{n-1,j}(a_1,\dots,a_s)-\gamma_n G_{n-1,j-1}(b_1,\dots,b_s)\right).$$

In general, we cannot expect (1.4) or (1.7) to happen, but as we will show in this paper, quite a few formulas for basic hypergeometric series can be proved in this way, such as Jackson's ${}_8\phi_7$ summation, Watson's ${}_8\phi_7$ transformation, Bailey's ${}_{10}\phi_9$ transformation, Singh's quadratic transformation, and a C_r extension of Jackson's ${}_8\phi_7$ sum due to Schlosser. The same method can also be used to prove a new finite form of Jacobi's triple product identity and a finite form of Watson's quintuple product identity.

Since all identities are trivial when $n=0$, we will only indicate the corresponding recurrence relations (1.4) (or, in addition, (1.7)) in our proofs.

It is worth noticing that (1.4) is not a special case of Sister Celine's method [15, p. 58, (4.3.1)] due to the change of parameters $a_i\to b_i$ in the right-hand side. For the same reason, Eq. (1.7) is not a hybrid of Zeilberger's method and Sister Celine's method.

2 Jackson's ${}_8\phi_7$ summation formula

Jackson [10] (see [7, Appendix (II.22)]) obtained a summation formula for a terminating ${}_8\phi_7$ series, which is one of the most powerful results in the theory and application of basic hypergeometric series.

Theorem 2.1 (Jackson's classical ${}_8\phi_7$ summation). *For $n\geq 0$, there holds*

$${}_8\phi_7\left[\begin{matrix} a,\ qa^{\frac12},\ -qa^{\frac12},\ b,\ c,\ d,\ e,\ q^{-n} \\ a^{\frac12},\ -a^{\frac12},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{n+1}\end{matrix};q,\ q\right]=\frac{(aq,aq/bc,aq/bd,aq/cd;q)_n}{(aq/b,aq/c,aq/d,aq/bcd;q)_n}, \tag{2.1}$$

where $a^2q^{n+1}=bcde$.

Proof. Let

$$F_{n,k}(a,b,c,d,q)=\frac{1-aq^{2k}}{1-a}\frac{(a,b,c,d,a^2q^{n+1}/bcd,q^{-n};q)_k}{(q,aq/b,aq/c,aq/d,bcdq^{-n}/a,aq^{n+1};q)_k}q^k$$

be the k-th term in Jackson's ${}_8\phi_7$ series. Applying the contiguous relation (1.1) with $a_r=a^2q^n/bcd$ and $a_{r+1}=q^{-n}$, we see that

$$F_{n,k}(a,b,c,d,q)-F_{n-1,k}(a,b,c,d,q)=\alpha_n F_{n-1,k-1}(aq^2,bq,cq,dq,q)$$

where

$$\alpha_n = \frac{(a^2q^n/bcd - q^{-n})(1-bcd/a)(1-aq)(1-aq^2)(1-b)(1-c)(1-d)q}{(1-bcdq^{-n}/a)(1-aq^n)(1-aq/b)(1-aq/c)(1-aq/d)(1-bcdq^{1-n}/a)(1-aq^{n+1})}.$$

∎

Remark. A different inductive proof of Jackson's ${}_8\phi_7$ summation formula can be found in [19, p. 95]. Substituting $e = a^2q^{n+1}/bcd$ into (2.1) and letting $d \to \infty$, we obtain Jackson's ${}_6\phi_5$ summation (see [7, Appendix (II.21)]):

$${}_6\phi_5\left[\begin{matrix} a,\, qa^{\frac12},\, -qa^{\frac12},\, b,\, c,\, q^{-n} \\ a^{\frac12},\, -a^{\frac12},\, aq/b,\, aq/c,\, aq^{n+1} \end{matrix};q,\, \frac{aq^{n+1}}{bc}\right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}. \tag{2.2}$$

3 Watson's ${}_8\phi_7$ transformation formula

Watson's ${}_8\phi_7$ transformation (see, for example, [7, Appendix (III.18)]) formula may be stated as:

Theorem 3.1 (Watson's classical q-Whipple transformation). *For $n \geq 0$, there holds*

$${}_8\phi_7\left[\begin{matrix} a,\, qa^{\frac12},\, -qa^{\frac12},\, b,\, c,\, d,\, e,\, q^{-n} \\ a^{\frac12},\, -a^{\frac12},\, aq/b,\, aq/c,\, aq/d,\, aq/e,\, aq^{n+1} \end{matrix};q,\, \frac{a^2q^{n+2}}{bcde}\right]$$
$$= \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n}\,{}_4\phi_3\left[\begin{matrix} aq/bc,\, d,\, e,\, q^{-n} \\ aq/b,\, aq/c,\, deq^{-n}/a, \end{matrix};q,\, q\right]. \tag{3.1}$$

Proof. Let

$$F_{n,k}(a,b,c,d,e,q) = F_k(a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, q^{-n}; q, a^2q^{n+2}/bcde).$$

Applying the contiguous relation (1.2) with $a_{r+1} = q^{-n}$ and $z = \frac{a^2q^{n+1}}{bcde}$, we have

$$F_{n,k}(a,b,c,d,e,q) - F_{n-1,k}(a,b,c,d,e,q) = \beta_n F_{n-1,k-1}(aq^2, bq, cq, dq, eq, q),$$

where

$$\beta_n = -\frac{(1-aq)(1-aq^2)(1-b)(1-c)(1-d)(1-e)a^2q^{n+1}}{(1-aq/b)(1-aq/c)(1-aq/d)(1-aq/e)(1-aq^n)(1-aq^{n+1})bcde}.$$

Let

$$G_{n,k}(a,b,c,d,e,q) = \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n}\frac{(aq/bc, d, e, q^{-n}; q)_k}{(q, aq/b, aq/c, deq^{-n}/a; q)_k}q^k.$$

Then $G_{n,k}(a,b,c,d,e,q)$ satisfies

$$G_{n,k}(a,b,c,d,e,q) - G_{n-1,k}(a,b,c,d,e,q) - \beta_n G_{n-1,k-1}(aq^2, bq, cq, dq, eq, q)$$
$$= H_{n,k} - H_{n,k-1},$$

where

$$H_{n,k} = \frac{(aq;q)_{n-1}(aq/de;q)_n}{(aq/d, aq/e; q)_n}\frac{(aq/bc, q^{1-n}; q)_k (d, e; q)_{k+1}}{(q, aq/b, aq/c; q)_k (deq^{-n}/a; q)_{k+1}}.$$

∎

Remark. The order of our recurrence relation is lower than that generated by q-Zeilberger's algorithm or the q-WZ method [12,15]. In fact, the q-Zeilberger's algorithm will generate a recursion of order 3 for the right-hand side of (3.1).

We can now obtain a transformation of terminating very-well-poised ${}_8\phi_7$ series [7, (2.10.3)]:

Corollary 3.2. *For $n \geq 0$, there holds*

$$
{}_8\phi_7\left[\begin{matrix} a,\, qa^{\frac12},\, -qa^{\frac12},\, b,\, c,\, d,\, e,\, q^{-n} \\ a^{\frac12},\, -a^{\frac12},\, aq/b,\, aq/c,\, aq/d,\, aq/e,\, aq^{n+1} \end{matrix};q,\, \frac{a^2q^{n+2}}{bcde}\right]
$$
$$
= \frac{(aq, \lambda q/e; q)_n}{(aq/e, \lambda q; q)_n} {}_8\phi_7\left[\begin{matrix} \lambda,\, q\lambda^{\frac12},\, -q\lambda^{\frac12},\, \lambda b/a,\, \lambda c/a,\, \lambda d/a,\, e,\, q^{-n} \\ \lambda^{\frac12},\, -\lambda^{\frac12},\, aq/b,\, aq/c,\, aq/d,\, \lambda q/e,\, \lambda q^{n+1} \end{matrix};q,\, \frac{aq^{n+1}}{e}\right], \tag{3.2}
$$

where $\lambda = a^2q/bcd$.

Proof. This is just the applications of (3.1) on both sides of (3.2). ∎

4 Bailey's ${}_{10}\phi_9$ transformation formula

In this section, we show that Bailey's ${}_{10}\phi_9$ transformation formula (see [7, Appendix (III.28)]) can also be proved by this method.

Theorem 4.1 (Bailey's classical ${}_{10}\phi_9$ transformation). *For $n \geq 0$, there holds*

$$
{}_{10}\phi_9\left[\begin{matrix} a,\, qa^{\frac12},\, -qa^{\frac12},\, b,\, c,\, d,\, e,\, f,\, \lambda aq^{n+1}/ef,\, q^{-n} \\ a^{\frac12},\, -a^{\frac12},\, aq/b,\, aq/c,\, aq/d,\, aq/e,\, aq/f,\, efq^{-n}/\lambda,\, aq^{n+1} \end{matrix};q,\, q\right]
$$
$$
= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q; q)_n}
$$
$$
\times\, {}_{10}\phi_9\left[\begin{matrix} \lambda,\, q\lambda^{\frac12},\, -q\lambda^{\frac12},\, \lambda b/a,\, \lambda c/a,\, \lambda d/a,\, e,\, f,\, \lambda aq^{n+1}/ef,\, q^{-n} \\ \lambda^{\frac12},\, -\lambda^{\frac12},\, aq/b,\, aq/c,\, aq/d,\, \lambda q/e,\, \lambda q/f,\, efq^{-n}/a,\, \lambda q^{n+1} \end{matrix};q,\, q\right], \tag{4.1}
$$

where $\lambda = a^2q/bcd$.

Proof. Note that both sides of (4.1) are very-well poised. Let

$$
F_{n,k}(a,b,c,d,e,f,q) = F_k(a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, q).
$$

Applying the contiguous relation (1.1) with $a_r = \lambda aq^n/ef$ and $a_{r+1} = q^{-n}$, we have

$$
F_{n,k}(a,b,c,d,e,f,q) - F_{n-1,k}(a,b,c,d,e,f,q) = \alpha_n F_{n-1,k-1}(aq^2, bq, cq, dq, eq, fq, q), \tag{4.2}
$$

where

$$\alpha_n = -\frac{(1-b)(1-c)(1-d)(1-e)(1-f)}{(1-aq/b)(1-aq/c)(1-aq/d)(1-aq/e)(1-aq/f)}$$
$$\times \frac{(1-aq)(1-aq^2)(1-ef/\lambda)(1-\lambda aq^{2n}/ef)}{(1-aq^n)(1-aq^{n+1})(1-efq^{1-n}/\lambda)(1-efq^{-n}/\lambda)q^{n-1}}.$$

Let

$$G_{n,k}(a,b,c,d,e,f,q) = \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q; q)_n}$$
$$\times F_k(\lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, q).$$

Then we may verify that

$$G_{n,k}(a,b,c,d,e,f,q) - G_{n-1,k}(a,b,c,d,e,f,q) - \alpha_n G_{n-1,k-1}(aq^2, bq, cq, dq, eq, fq, q)$$
$$= H_{n,k} - H_{n,k-1},$$

where

$$H_{n,k} = \frac{(1-a\lambda q^{2n}/ef)(aq, \lambda q/e, \lambda q/f; q)_{n-1}(aq/ef; q)_n}{(aq/e, aq/f, \lambda q/ef, \lambda; q)_n}$$
$$\times \frac{(1-\lambda q^k/a)(\lambda b/a, \lambda c/a, \lambda d/a, \lambda aq^{n+1}/ef, q^{1-n}; q)_k (\lambda, e, f; q)_{k+1}}{(q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f; q)_k (efq^{-n}/a, \lambda q^n; q)_{k+1}}.$$ ∎

5 Singh's quadratic transformation

The following quadratic transformation (see [7, Appendix (III.21)]) was first proved by Singh [18]. For a more recent proof, see Askey and Wilson [4]. We will show that it can also be proved by induction in the same vein as in the previous sections.

Theorem 5.1 (Singh's quadratic transformation). *There holds*

$${}_4\phi_3\left[\begin{matrix} a^2,\ b^2,\ c,\ d \\ abq^{\frac{1}{2}},\ -abq^{\frac{1}{2}},\ -cd \end{matrix}; q,\ q\right] = {}_4\phi_3\left[\begin{matrix} a^2,\ b^2,\ c^2,\ d^2 \\ a^2b^2q,\ -cd,\ -cdq \end{matrix}; q^2,\ q^2\right], \tag{5.1}$$

provided that both series terminate.

Proof. Let $d = q^{-n}$. Then (5.1) may be written as

$${}_4\phi_3\left[\begin{matrix} a^2,\ b^2,\ c,\ q^{-n} \\ abq^{\frac{1}{2}},\ -abq^{\frac{1}{2}},\ -cq^{-n} \end{matrix}; q,\ q\right] = {}_4\phi_3\left[\begin{matrix} a^2,\ b^2,\ c^2,\ q^{-2n} \\ a^2b^2q,\ -cq^{-n},\ -cq^{1-n} \end{matrix}; q^2,\ q^2\right].$$

Let

$$F_{n,k}(a,b,c,q) = \frac{(a^2, b^2, c, q^{-n}; q)_k}{(q, abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -cq^{-n}; q)_k} q^k.$$

Applying the relation:

$$1-\frac{(1-xy)(1+cy)}{(1-y)(1+cxy)}=-\frac{(1+c)(1-x)y}{(1-y)(1+cxy)},$$

with $x=q^k$ and $y=q^{-n}$, one sees that

$$\begin{aligned}&F_{n,k}(a,b,c,q)-F_{n-1,k}(a,b,c,q)\\&\quad=-\frac{(1-a^2)(1-b^2)(1-c^2)q^{1-n}}{(1-a^2b^2q)(1+cq^{-n})(1+cq^{1-n})}F_{n-1,k-1}(aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq,q).\end{aligned}\tag{5.2}$$

Replacing n by $n-1$ in (5.2), we get

$$\begin{aligned}&F_{n-1,k}(a,b,c,q)-F_{n-2,k}(a,b,c,q)\\&\quad=-\frac{(1-a^2)(1-b^2)(1-c^2)q^{2-n}}{(1-a^2b^2q)(1+cq^{1-n})(1+cq^{2-n})}F_{n-2,k-1}(aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq,q).\end{aligned}\tag{5.3}$$

From (5.2) and (5.3) it follows that

$$\begin{aligned}&q(1+cq^{-n})[F_{n,k}(a,b,c,q)-F_{n-1,k}(a,b,c,q)]\\&-(1+cq^{2-n})[F_{n-1,k}(a,b,c,q)-F_{n-2,k}(a,b,c,q)]\\&=-\frac{(1-a^2)(1-b^2)(1-c^2)q^{2-n}}{(1-a^2b^2q)(1+cq^{1-n})}[F_{n-1,k-1}(aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq,q)-F_{n-2,k-1}(aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq,q)].\end{aligned}\tag{5.4}$$

If we apply (5.3) with a,b,c,k replaced by $aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq,k-1$, respectively, then (5.4), after dividing both sides by $q(1+cq^{-n})$, may be written as

$$F_{n,k}(a,b,c,q)-\alpha_nF_{n-1,k}(a,b,c,q)+\beta_nF_{n-2,k}(a,b,c,q)=\gamma_nF_{n-2,k-2}(aq,bq,cq^2,q),$$

where

$$\begin{aligned}\alpha_n&=\frac{(1+q)(1+cq^{1-n})}{q(1+cq^{-n})},\\\beta_n&=\frac{1+cq^{2-n}}{q(1+cq^{-n})},\\\gamma_n&=\frac{(1-a^2)(1-b^2)(1-c^2)(1-a^2q)(1-b^2q)(1-c^2q^2)q^{3-2n}}{(1-a^2b^2q)(1-a^2b^2q^3)(1+cq^{-n})(1+cq^{1-n})(1+cq^{2-n})(1+cq^{3-n})}.\end{aligned}$$

Let

$$G_{n,k}(a,b,c,q)=\frac{(a^2,b^2,c^2,q^{-2n};q^2)_k}{(q^2;a^2b^2q;q^2)_k(-cq^{-n};q)_{2k}}q^{2k}.$$

Then it is easy to verify that

$$G_{n,k}(a,b,c,q)-\alpha_n G_{n-1,k}(a,b,c,q)+\beta_n G_{n-2,k}(a,b,c,q)-\gamma_n G_{n-2,k-2}(aq,bq,cq^2,q)$$
$$=H_{n,k}-H_{n,k-1},$$

where

$$H_{n,k}=-\frac{(1-q^{2k-1})(a^2,b^2;q^2)_k(q^{4-2n};q^2)_{k-1}(c^2;q^2)_{k+1}q^{2-2n}}{(q^2;q^2)_{k-1}(a^2b^2q;q^2)_k(-cq^{-n};q)_{2k+2}}. \qquad \blacksquare$$

6 Schlosser's C_r extension of Jackson's ${}_8\phi_7$ summation formula

Based on a determinant formula of Krattenthaler [13, Lemma 34], Schlosser [16, 17] established a C_r extension of Jackson's ${}_8\phi_7$ summation formula.

Theorem 6.1 (Schlosser's C_r Jackson's sum). *Let $x_1,\ldots,x_r$, a, b, c and d be indeterminates and let n be a nonnegative integer. Suppose that none of the denominators in* (6.1) *vanish. Then*

$$\sum_{k_1,\ldots,k_r=0}^{n}\prod_{1\le i<j\le r}\frac{(x_iq^{k_i}-x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)}\prod_{i=1}^{r}\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}$$
$$\times\prod_{i=1}^{r}\frac{(ax_i^2,bx_i,cx_i,dx_i,a^2x_iq^{n-r+2}/bcd,q^{-n};q)_{k_i}q^{k_i}}{(q,ax_iq/b,ax_iq/c,ax_iq/d,bcdx_iq^{r-n-1}/a,ax_i^2q^{n+1};q)_{k_i}}$$
$$=\prod_{1\le i<j\le r}\frac{1-ax_ix_jq^n}{1-ax_ix_j}\prod_{i=1}^{r}\frac{(ax_i^2q,aq^{2-i}/bc,aq^{2-i}/bd,aq^{2-i}/cd;q)_n}{(aq^{2-r}/bcdx_i,ax_iq/b,ax_iq/c,ax_iq/d;q)_n}. \qquad (6.1)$$

We shall give an inductive proof of Schlosser's C_r extension of Jackson's ${}_8\phi_7$ summation formula. We first give a simple proof of the $n=1$ case of (6.1), which we state as the following lemma.

Lemma 6.2. *For $r\ge 0$, there holds*

$$\sum_{s_1,\ldots,s_r=0}^{1}\prod_{1\le i<j\le r}\frac{(x_iq^{s_i}-x_jq^{s_j})(1-ax_ix_jq^{s_i+s_j})}{(x_i-x_j)(1-ax_ix_jq)}\prod_{i=1}^{r}\frac{(-1)^{s_i}(bx_i,cx_i,dx_i,a^2x_iq^{3-r}/bcd;q)_{s_i}}{(ax_iq/b,ax_iq/c,ax_iq/d,bcdx_iq^{r-2}/a;q)_{s_i}}$$
$$=\prod_{i=1}^{r}\frac{(ax_i^2q,aq^{2-i}/bc,aq^{2-i}/bd,aq^{2-i}/cd;q)_1}{(aq^{2-r}/bcdx_i,ax_iq/b,ax_iq/c,ax_iq/d;q)_1}. \qquad (6.2)$$

Proof. Multiplying both sides by $\prod_{1\le i<j\le r}(x_i-x_j)(1-ax_ix_jq)$, Equation (6.2) may be written as

$$
\begin{aligned}
&\sum_{s_1,\dots,s_r=0}^{1}\prod_{1\le i<j\le r}(x_iq^{s_i}-x_jq^{s_j})(1-ax_ix_jq^{s_i+s_j})\\
&\quad\times\prod_{i=1}^{r}(-1)^{s_i}(bx_i,cx_i,dx_i,a^2x_iq^{3-r}/bcd;q)_{s_i}(ax_iq/b,ax_iq/c,ax_iq/d,bcdx_iq^{r-2}/a;q)_{1-s_i}\\
&\quad=\prod_{1\le i<j\le r}(x_i-x_j)(1-ax_ix_jq)\prod_{i=1}^{r}\frac{-bcdx_i(ax_i^2q,aq^{2-i}/bc,aq^{2-i}/bd,aq^{2-i}/cd;q)_1}{aq^{2-r}}.
\end{aligned}
\tag{6.3}
$$

Denote the left-hand side of (6.3) by L. If $x_i=x_j$ or $x_i=0$ for some $1\le i,j\le r$ $(i\ne j)$, then it is easily seen that L is equal to 0.

If $ax_ix_jq=1$ for some $i\ne j$, then for $0\le s_i,s_j\le 1$, we have

$$
\begin{aligned}
1-ax_ix_jq^{s_i+s_j}&=1-q^{s_i+s_j-1},\\
(bx_i,cx_i,dx_i,a^2x_iq^{3-r}/bcd;q)_{s_i}&=\frac{(bx_i,cx_i,dx_i,bcdx_ix_j^2q^{r-1};q)_{s_i}}{(-bcdx_ix_j^2q^{r-1})^{s_i}},\\
(ax_iq/b,ax_iq/c,ax_iq/d,bcdx_iq^{r-2}/a;q)_{1-s_i}&=\frac{(bx_j,cx_j,dx_j,bcdx_i^2x_jq^{r-1};q)_{1-s_i}}{(-bcdx_j^3)^{1-s_i}}.
\end{aligned}
$$

It follows that each term on the left-hand side of (6.3) subject to $s_i+s_j=1$ is equal to 0. Besides, any two terms cancel each other if they have the same s_k except for $s_i=s_j=0$ and $s_i=s_j=1$, respectively. Therefore, L is equal to 0.

If $ax_i^2q=1$ for some $1\le i\le r$, then we observe that any two different terms on the left-hand side of (6.3) with the same s_k $(k\ne i)$ cancel each other, and hence L is equal to 0.

Summarizing the above cases, we see that L is divisible by

$$
\prod_{1\le i<j\le r}(x_i-x_j)(1-ax_ix_jq)\prod_{i=1}^{r}x_i(1-ax_i^2q).
$$

Now we consider the left-hand side of (6.3) as a polynomial in x_r. It is easy to see that the coefficient of x_r^{2r+2} is given by

$$
\begin{aligned}
&\sum_{s_1,\dots,s_r=0}^{1}\left(\prod_{1\le i<j\le r-1}(x_iq^{s_i}-x_jq^{s_j})\prod_{i=1}^{r-1}(ax_iq^{s_i})\right)(-1)^{s_r}q^{(2r-2)s_r}(a^2q^{3-r})^{s_r}(a^2q^{r+1})^{1-s_r}\\
&\quad\times\prod_{i=1}^{r-1}(-1)^{s_i}(bx_i,cx_i,dx_i,a^2x_iq^{3-r}/bcd;q)_{s_i}(ax_iq/b,ax_iq/c,ax_iq/d,bcdx_iq^{r-2}/a;q)_{1-s_i}\\
&\quad=0.
\end{aligned}
$$

Therefore, if we write

$$L = P \prod_{1\le i<j\le r} (x_i - x_j)(1 - ax_ix_jq) \prod_{i=1}^{r} x_i(1 - ax_i^2 q),$$

then P is independent of x_r. By symmetry, one sees that P is independent of all x_i $(1 \le i \le r)$. Now, taking $x_i = q^{i-1}$ $(1 \le i \le r)$, then

$$\prod_{1\le i<j\le r} (x_i q^{s_i} - x_j q^{s_j}) = \prod_{1\le i<j\le r} (q^{i+s_i-1} - q^{j+s_j-1}),$$

which is equal to 0 unless $s_1 \le s_2 \le \cdots \le s_r$. It follows that

$$\begin{aligned} P = &\sum_{0\le s_1\le\cdots\le s_r\le 1} \prod_{1\le i<j\le r} \frac{(q^{i+s_i-1} - q^{j+s_j-1})(1 - aq^{i+j+s_i+s_j-2})}{(q^{i-1} - q^{j-1})(1 - aq^{i+j-1})} \prod_{i=1}^{r} \frac{(-1)^{s_i}}{q^{i-1}(1 - aq^{2i-1})} \\ &\times \prod_{i=1}^{r} (bq^{i-1}, cq^{i-1}, dq^{i-1}, a^2q^{i+2-r}/bcd; q)_{s_i} (aq^i/b, aq^i/c, aq^i/d, bcdq^{i+r-3}/a; q)_{1-s_i} \\ = &\sum_{k=0}^{r} (-1)^{r-k} q^{\frac{k(k+1-2r)}{2}} \begin{bmatrix} r \\ k \end{bmatrix} \frac{1 - aq^{2k}}{(aq^k; q)_{r+1}} (bq^k, cq^k, dq^k, a^2q^{3-r+k}/bcd; q)_{r-k} \\ &\times (aq/b, aq/c, aq/d, bcdq^{r-2}/a; q)_k, \end{aligned}$$

where we have assumed $s_k = 0$ and $s_{k+1} = 1$. It remains to show that

$$P = (-bcdq^{r-2}/a)^r (aq^{2-r}/bc, aq^{2-r}/bd, aq^{2-r}/cd; q)_r,$$

or

$$\begin{aligned} &\sum_{k=0}^{r} (-1)^k q^{\frac{k(k+1-2r)}{2}} \begin{bmatrix} r \\ k \end{bmatrix} \frac{1 - aq^{2k}}{(aq^k; q)_{r+1}} \frac{(aq/b, aq/c, aq/d, bcdq^{r-2}/a; q)_k}{(b, c, d, a^2q^{3-r}/bcd; q)_k} \\ &\quad = \frac{(bcdq^{r-2}/a)^r (aq^{2-r}/bc, aq^{2-r}/bd, aq^{2-r}/cd; q)_r}{(b, c, d, a^2q^{3-r}/bcd; q)_r}. \end{aligned} \tag{6.4}$$

But (6.4) follows easily from Jackson's ${}_8\phi_7$ summation formula (2.1) with parameter substitutions $b \mapsto aq/b$, $c \mapsto aq/c$, $d \mapsto aq/d$, and $n \mapsto r$. This proves (6.2). ∎

Proof of Theorem 6.1. Suppose (6.1) holds for n. It is easy to verify that

$$\begin{aligned} &\frac{(1 - a^2x_iq^{n+k_i-r+2}/bcd)(1 - bcdx_iq^{k_i+r-n-2})}{(1 - q^{-n+k_i-1})(1 - ax_i^2q^{n+k_i+1})} \\ &= -\frac{q^{n+1}(1 - a^2x_iq^{n-r+2}/bcd)(1 - bcdx_iq^{r-n-2}/a)}{(1 - q^{n+1})(1 - ax_i^2q^{n+1})} \\ &\quad + \frac{(1 - a^2x_iq^{2n-r+3}/bcd)(1 - bcdx_iq^{r-1}/a)(1 - ax_i^2q^{k_i})(1 - q^{k_i})}{(1 - q^{n+1})(1 - ax_i^2q^{n+1})(1 - q^{-n+k_i-1})(1 - ax_i^2q^{n+k_i+1})}. \end{aligned}$$

Hence,

$$\prod_{i=1}^{r}\frac{(1-a^2x_iq^{n+k_i-r+2}/bcd)(1-bcdx_iq^{k_i+r-n-2})}{(1-q^{-n+k_i-1})(1-ax_i^2q^{n+k_i+1})}=\sum_{s_1,\dots,s_r=0}^{1}\frac{\alpha_{s_1,\dots,s_r}(1-ax_i^2q^{k_i})^{s_i}(1-q^{k_i})^{s_i}}{(1-q^{-n+k_i-1})^{s_i}(1-ax_i^2q^{n+k_i+1})^{s_i}},$$

where

$$\alpha_{s_1,\dots,s_r}=\frac{1}{(1-q^{n+1})^r}\prod_{i=1}^{r}\frac{(-q^{n+1})^{1-s_i}}{1-ax_i^2q^{n+1}}(1-a^2x_iq^{n-r+2}/bcd)^{1-s_i}$$
$$\times(1-bcdx_iq^{r-n-2}/a)^{1-s_i}(1-a^2x_iq^{2n-r+3}/bcd)^{s_i}(1-bcdx_iq^{r-1}/a)^{s_i}.$$

It follows that

$$\prod_{1\le i<j\le r}\frac{(x_iq^{k_i}-x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})}{(x_i-x_j)(1-ax_ix_j)}\prod_{i=1}^{r}\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}$$
$$\times\prod_{i=1}^{r}\frac{(ax_i^2,bx_i,cx_i,dx_i,a^2x_iq^{n-r+3}/bcd,q^{-n-1};q)_{k_i}q^{k_i}}{(q,ax_iq/b,ax_iq/c,ax_iq/d,bcdx_iq^{r-n-2}/a,ax_i^2q^{n+2};q)_{k_i}}$$
$$=\sum_{s_1,\dots,s_r=0}^{1}\beta_{s_1,\dots,s_r}\prod_{1\le i<j\le r}\frac{(x_iq^{k_i}-x_jq^{k_j})(1-ax_ix_jq^{k_i+k_j})}{(x_iq^{s_i}-x_jq^{s_j})(1-ax_ix_jq^{s_i+s_j})}\prod_{i=1}^{r}\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2q^{2s_i}}$$
$$\times\prod_{i=1}^{r}\frac{(ax_i^2q^{2s_i},bx_iq^{s_i},cx_iq^{s_i},dx_iq^{s_i},a^2x_iq^{n-r+s_i+2}/bcd,q^{-n};q)_{k_i-s_i}q^{k_i-s_i}}{(q,ax_iq^{s_i+1}/b,ax_iq^{s_i+1}/c,ax_iq^{s_i+1}/d,bcdx_iq^{s_i+r-n-1}/a,ax_i^2q^{n+2s_i+1};q)_{k_i-s_i}},\tag{6.5}$$

where

$$\beta_{s_1,\dots,s_r}=\alpha_{s_1,\dots,s_r}\prod_{1\le i<j\le r}\frac{(x_iq^{s_i}-x_jq^{s_j})(1-ax_ix_jq^{s_i+s_j})}{(x_i-x_j)(1-ax_ix_j)}\prod_{i=1}^{r}\frac{1-ax_i^2q^{2s_i}}{1-ax_i^2}$$
$$\times\frac{(ax_i^2;q)_{2s_i}(bx_i,cx_i,dx_i;q)_{s_i}q^{s_i}(1-ax_i^2q^{n+s_i+1})^{1-2s_i}(1-q^{-n-1})}{(ax_iq/b,ax_iq/c,ax_iq/d;q)_{s_i}(bcdx_iq^{r-n-2}/a;q)_{s_i+1}(a^2x_iq^{n-r+2}/bcd;q)_{1-s_i}}$$
$$=\prod_{1\le i<j\le r}\frac{(x_iq^{s_i}-x_jq^{s_j})(1-ax_ix_jq^{s_i+s_j})}{(x_i-x_j)(1-ax_ix_j)}$$
$$\times\prod_{i=1}^{r}\frac{(-1)^{s_i}(ax_i^2q;q)_{2s_i}(bx_i,cx_i,dx_i,bcdx_iq^{r-1}/a,a^2x_iq^{2n-r+3}/bcd,q)_{s_i}}{q^{ns_i}(ax_i^2q^{n+1},bcdx_iq^{r-n-2}/a;q)_{2s_i}(ax_iq/b,ax_iq/c,ax_iq/d;q)_{s_i}}.$$

By the induction hypothesis, the right-hand side of (6.5) is equal to

$$
\begin{aligned}
&\sum_{s_1,\ldots,s_r=0}^{1} \beta_{s_1,\ldots,s_r} \prod_{1\le i<j\le r} \frac{1-ax_ix_jq^{n+s_i+s_j}}{1-ax_ix_jq^{s_i+s_j}} \\
&\quad\times \prod_{i=1}^{r} \frac{(ax_i^2q^{2s_i+1}, aq^{2-i}/bc, aq^{2-i}/bd, aq^{2-i}/cd; q)_n}{(aq^{2-r-s_i}/bcdx_i, ax_iq^{s_i+1}/b, ax_iq^{s_i+1}/c, ax_iq^{s_i+1}/d; q)_n} \\
&= \sum_{s_1,\ldots,s_r=0}^{1} \prod_{1\le i<j\le r} \frac{(x_iq^{s_i}-x_jq^{s_j})(1-ax_ix_jq^{n+s_i+s_j})}{(x_i-x_j)(1-ax_ix_j)} \\
&\quad\times \prod_{i=1}^{r} \frac{(bx_i, cx_i, dx_i, a^2x_iq^{2n-r+3}/bcd; q)_{s_i}(ax_i^2q, aq^{2-i}/bc, aq^{2-i}/bd, aq^{2-i}/cd; q)_n}{(bcdx_iq^{r-n-2})^{s_i}(aq^{2-r}/bcdx_i, ax_iq/b, ax_iq/c, ax_iq/d; q)_{n+s_i}}. \qquad (6.6)
\end{aligned}
$$

Replacing a by aq^n in (6.2), we have

$$
\begin{aligned}
&\sum_{s_1,\ldots,s_r=0}^{1} \prod_{1\le i<j\le r} \frac{(x_iq^{s_i}-x_jq^{s_j})(1-ax_ix_jq^{n+s_i+s_j})}{(x_i-x_j)(1-ax_ix_j)} \\
&\quad\times \prod_{i=1}^{r} \frac{(bx_i, cx_i, dx_i, a^2x_iq^{3-r}/bcd; q)_{s_i}(-1)^{s_i}}{(ax_iq^{n+1}/b, ax_iq^{n+1}/c, ax_iq^{n+1}/d, bcdx_iq^{r-n-2}/a; q)_{s_i}} \\
&\qquad= \prod_{1\le i<j\le r} \frac{1-ax_ix_jq^{n+1}}{1-ax_ix_j} \prod_{i=1}^{r} \frac{(ax_i^2q^{n+1}, aq^{n+2-i}/bc, aq^{n+2-i}/bd, aq^{n+2-i}/cd; q)_1}{(aq^{n+2-r}/bcdx_i, ax_iq^{n+1}/b, ax_iq^{n+1}/c, ax_iq^{n+1}/d; q)_1}.
\end{aligned}
$$

Therefore, the right-hand side of (6.6) is equal to

$$
\prod_{1\le i<j\le r} \frac{1-ax_ix_jq^{n+1}}{1-ax_ix_j} \prod_{i=1}^{r} \frac{(ax_i^2q, aq^{2-i}/bc, aq^{2-i}/bd, aq^{2-i}/cd; q)_{n+1}}{(aq^{2-r}/bcdx_i, ax_iq/b, ax_iq/c, ax_iq/d; q)_{n+1}}.
$$

Namely, formula (6.1) holds for $n+1$. This completes the inductive step, and we conclude that (6.1) holds for all integers $n \ge 0$. ∎

We end this section with two new allied identities.

Proposition 6.3. *For $n \geq 0$, we have*

$$\sum_{s_1,\dots,s_r=0}^{n} \prod_{1\le i<j\le r} \frac{(x_i q^{s_i} - x_j q^{s_j})(1 - a x_i x_j q^{s_i+s_j})}{(x_i - x_j)(1 - a x_i x_j q^n)} \prod_{i=1}^{r} \frac{1}{q^{(r-1)s_i}} = \frac{(n+1)(q^{n+1};q^{n+1})_{r-1}}{q^{n\binom{r}{2}}(q;q)_{r-1}}, \tag{6.7}$$

$$\sum_{s_1,\dots,s_r=0}^{n} \prod_{1\le i<j\le r} \frac{(x_i q^{s_i} - x_j q^{s_j})(1 - a x_i x_j q^{s_i+s_j})}{(x_i - x_j)(1 - a x_i x_j q^n)} \prod_{i=1}^{r} \frac{1}{(-q)^{(r-1)s_i}}$$
$$= \begin{cases} \dfrac{(-q^{n+1};q^{n+1})_{r-1}}{q^{n\binom{r}{2}}(-q;q)_{r-1}}, & \text{if } n \equiv 0 \pmod 2, \\ 0, & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{6.8}$$

Proof. Assume $|q| < 1$. Similarly to the proof of (6.2), we can show that the left-hand side of (6.7) is independent of $x_1, \dots, x_r$. In particular, taking $x_i = q^{mi}$ $(1 \le i \le r)$ and letting $m \to +\infty$, the left-hand side of (6.7) then becomes

$$\sum_{s_1,\dots,s_r=0}^{n} \prod_{i=1}^{r} \frac{1}{q^{(i-1)s_i}} = \prod_{i=1}^{r} \sum_{s_i=0}^{n} \frac{1}{q^{(i-1)s_i}},$$

which is clearly equal to the right-hand side of (6.7). Similarly we can prove (6.8). ∎

7 A finite form of Lebesgue's identity and Jacobi's triple product identity

Using the same vein of (1.4), we can also derive some other identities. Here we will prove a finite form of Lebesgue's identity and Jacobi's triple product identity. As usual, the *q-binomial coefficients* are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad n, k \in \mathbb{Z}.$$

Theorem 7.1 (A finite form of Lebesgue's and Jacobi's identities). *For $n \geq 0$, there holds*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k+1)/2}}{(aq^k;q)_{n+1}} = \frac{(-q;q)_n}{(a;q^2)_{n+1}}. \tag{7.1}$$

Proof. Let

$$F_{n,k}(a,q) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k+1)/2}}{(aq^k;q)_{n+1}}.$$

Noticing the trivial relation

$$\begin{bmatrix} n \\ k \end{bmatrix}(1-aq^n) = \begin{bmatrix} n-1 \\ k \end{bmatrix}(1-aq^{n+k}) + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}(1-aq^k)q^{n-k}, \tag{7.2}$$

we have

$$F_{n,k}(a,q) = \frac{1}{1-aq^n}F_{n-1,k}(a,q) + \frac{q^n}{1-aq^n}F_{n-1,k-1}(aq^2,q). \qquad \blacksquare$$

Letting $n \to \infty$ in (7.1), we immediately get Lebesgue's identity:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} q^{k(k+1)/2} = (aq;q^2)_\infty(-q;q)_\infty.$$

It is interesting to note that (7.1) is also a finite form of *Jacobi's triple product identity*

$$\sum_{k=-\infty}^{\infty} q^{k^2} z^k = (q^2, -q/z, -qz; q^2)_\infty. \tag{7.3}$$

Indeed, performing parameter replacements $n \to m+n$, $a \to -zq^{-2m}$ and $k \to k+m$ in (7.1), and noticing the following relation

$$\frac{(-zq^{-2m};q^2)_{m+n+1}}{(-zq^{k-m};q)_{m+n+1}} q^{\binom{m+k+1}{2}} = \frac{(-q^2/z;q^2)_m(-z;q^2)_{n+1}}{(-q/z;q)_{m-k}(-z;q)_{n+k+1}} q^{k^2} z^k,$$

we obtain

$$\sum_{k=-m}^{n} \begin{bmatrix} m+n \\ m+k \end{bmatrix} \frac{(-q^2/z;q^2)_m(-z;q^2)_{n+1}}{(-q/z;q)_{m-k}(-z;q)_{n+k+1}} q^{k^2} z^k = (-q;q)_{m+n}. \tag{7.4}$$

Letting $m, n \to \infty$ in (7.4) and applying the relation

$$\frac{(-q,q,-q/z,-z;q)_\infty}{(-q^2/z,-z;q^2)_\infty} = (q^2,-q/z,-qz;q^2)_\infty,$$

we are led to (7.3).

Remark. Theorem 7.1 is in fact the $b = q^{-n}$ case of the q-analogue of Kummer's theorem (see [7, Appendix (II.9)]):

$${}_2\phi_1\left[\begin{matrix} a,\ b \\ q,\ aq/b, \end{matrix}\ ; q,\ -q/b\right] = \frac{(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}.$$

There also exists another finite form of *Lebesgue's identity* as follows:

$$\sum_{k=0}^{n} \frac{(a;q)_k(q^{-2n};q^2)_k}{(q;q)_k(q^{-2n};q)_k} q^k = \frac{(aq;q^2)_n}{(q;q^2)_n}, \tag{7.5}$$

which is the special case $b=0$ of the following identity due to Andrews-Jain (see [3,8,11]):

$$\sum_{k=0}^{n} \frac{(a,b;q)_k(q^{-2n};q^2)_k q^k}{(q;q)_k(abq;q^2)_k(q^{-2n};q)_k} = \frac{(aq,bq;q^2)_n}{(q,abq;q^2)_n}.$$

A bijective proof of Lebesgue's identity was obtained by Bessenrodt [5] using Sylvester's bijection. It would be interesting to find a bijective proof of (7.1) and (7.5). Alladi and Berkovich [2] have given different finite analogues of Jacobi's triple product and Lebesgue's identities.

8 A finite form of Watson's quintuple product identity

Watson's quintuple product identity (see [7, p. 147]) states that

$$\sum_{k=-\infty}^{\infty} (z^2q^{2k+1}-1)z^{3k+1}q^{k(3k+1)/2} = (q,z,q/z;q)_\infty(qz^2,q/z^2;q^2)_\infty. \tag{8.1}$$

We will show that (8.1) follows from the following trivial identity.

Theorem 8.1 (A finite form of Watson's quintuple product identity). *For $n \geq 0$, there holds*

$$\sum_{k=0}^{n}(1-z^2q^{2k+1})\begin{bmatrix} n \\ k \end{bmatrix}\frac{(zq;q)_n}{(z^2q^{k+1};q)_{n+1}}z^kq^{k^2} = 1. \tag{8.2}$$

Proof. Let

$$F_{n,k}(z,q) = (1-z^2q^{2k+1})\begin{bmatrix} n \\ k \end{bmatrix}\frac{(zq;q)_n}{(z^2q^{k+1};q)_{n+1}}z^kq^{k^2}.$$

Replacing a by z^2q in (7.2), one sees that

$$F_{n,k}(z,q) = \frac{1-zq^n}{1-z^2q^{n+1}}F_{n-1,k}(z,q) + \frac{(1-zq)zq^n}{1-z^2q^{n+1}}F_{n-1,k-1}(zq,q). \qquad \blacksquare$$

Making the substitutions $n \to m+n$, $z \to -zq^{-m}$ and $k \to m+k$ in (8.2), and noticing the following relation

$$\frac{(-zq^{1-m};q)_{m+n}}{(z^2q^{k-m+1};q)_{m+n+1}}(-zq^{-m})^kq^{(k+m)^2} = \frac{(-q/z;q)_{m-1}(-z;q)_{n+1}}{(1/z^2;q)_{m-k}(z^2q;q)_{n+k+1}}z^{3k-1}q^{k(3k+1)/2},$$

we obtain

$$\sum_{k=-m}^{n}(1-z^2q^{2k+1})\begin{bmatrix} m+n \\ m+k \end{bmatrix}\frac{(-q/z;q)_{m-1}(-z;q)_{n+1}}{(1/z^2;q)_{m-k}(z^2q;q)_{n+k+1}}z^{3k-1}q^{k(3k+1)/2} = 1. \tag{8.3}$$

Letting $m, n \to \infty$ and applying the relation

$$\frac{(-z;q)_\infty(-q/z;q)_\infty}{(z^2q;q)_\infty(1/z^2;q)_\infty} = \frac{-z^2}{(qz^2;q^2)_\infty(z;q)_\infty(q/z^2;q)_\infty(q/z;q)_\infty},$$

we immediately obtain Watson's quintuple product identity (8.1).

Remark. Note that (8.2) is the limiting case $c \to \infty$ of (2.2) with $a = z^2q$ and $b = zq$. Paule [14] has proved the $m = n$ case of (8.3). On the other hand, Chen et al. [6] obtained the following finite form of Watson's quintuple product identity:

$$\sum_{k=0}^{n}(1+zq^k)\begin{bmatrix} n \\ k \end{bmatrix}\frac{(z;q)_{n+1}}{(z^2q^k;q)_{n+1}}z^kq^{k^2} = 1. \tag{8.4}$$

Finally, by letting $n \to \infty$ and substituting $z \to z/q$ and $q \to q^2$ in (8.2), and $z \to zq$ and $q \to q^2$ in (8.4), respectively, we get,

$$1 + \sum_{k=1}^{\infty}\frac{z^kq^{2k^2-k}(z^2q^2;q^2)_{k-1}(1-z^2q^{4k})}{(q^2;q^2)_k} = (-zq;q^2)_\infty(z^2q^4;q^4)_\infty, \tag{8.5}$$

$$\sum_{k=0}^{\infty}\frac{z^kq^{2k^2+k}(z^2q^2;q^2)_k(1+zq^{2k+1})}{(q^2;q^2)_k} = (-zq;q^2)_\infty(z^2q^4;q^4)_\infty. \tag{8.6}$$

Alladi and Berkovich [1] have given the partition interpretations of (8.5) and (8.6). It would be interesting to find the corresponding combinatorial interpretations of (8.2) and (8.4).

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